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N.E. Tomova

Natural Implication and Modus Ponens Principle[1]

Tomova Natalya Evgenyevna
Department of Logic, Institute of Philosophy, Russian Academy of Sciences.
Volkhonka 14/5, Moscow, 119991, Russian Federation.
e-mail: natalya-tomova@yandex.ru

In [6] the definition of *natural* implication was introduced. One of the criteria for *natural* implication is the normality of logical matrix [2, p. 134], a condition sufficient for verification of *modus ponens*. In this paper two definitions of *modus ponens* are regarded: in the designation-preserving sense and in the tautologousness-preserving sense. These formulations are considered as applied to two-valued and three-valued cases. In two-valued case these formulations are equivalent. But in case of three-valued logic we have another situation: they are not equivalent, but the first formulation entails the second, the reverse is not the case. According to that fact, the definition of *natural* implication is transformed and truth tables for extended class of *natural* implications are presented.

Keywords: three-valued logic, natural implication, modus ponens principle

1 Introduction

In paper [6] (see also [7]) we introduced the definition of natural implication, which defines the class of 28 implications with some useful properties. Among them there are the implications of best-known three-valued logics.

One of the criteria for natural implication is the normality of logical matrix [2, p. 134], a condition sufficient for verification of *modus ponens*. In other words an implication should be required to preserve the designated value. In this paper we consider the weakening of that condition and extended class of natural implications will be regarded.

2 Two formulations of modus ponens

In [5, p. 70] N. Rescher pointed out the necessity “to distiguish between two ways in which a modus ponens principle can be operative in a system of many-valued logic”. There are two different formulations:

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(i) a *stronger* condition: whenever p and $p \to q$ are both designated truth-values, then q must also be a designated value;

(ii) a *weaker* condition: whenever p and $p \to q$ are both tautologies, then q must also be a tautology.

We would like to stress the distinction between these two formulations and give it in symbolic form. Let us define some related notions and notations.

DEFINITION 1. The language $L_{\to}$ is a propositional language with the following alphabet: $p, q, r, \ldots$ — propositional variables; $\to$ — binary logical connective; $(,)$ — technical symbols.

DEFINITION 2. A definition of $L_{\to}$-formula is as usual:

(1) if A is propositional variable, then A is $L_{\to}$-formula;

(2) if A and B are $L_{\to}$-formulas, then $A \to B$ is $L_{\to}$-formula;

(3) nothing else is $L_{\to}$-formula.

DEFINITION 3. A logical matrix is a structure $\mathfrak{M} =< V, F, D >$, where V is the set of truth-values, F is a set of functions on V called *basic functions*, and D is a set of designated values, D is a subset of V.

DEFINITION 4. A valuation v of an arbitrary $L_{\to}$-formula A in $\mathfrak{M}$ (symbolically — $|A|_v^{\mathfrak{M}}$) is defined as usual: $|p|_v^{\mathfrak{M}} \in V$, if p is a propositional variable; if A and B are $L_{\to}$-formulas, and $\to$ is basic function in $\mathfrak{M}$, then $|A \to B|_v^{\mathfrak{M}} = |A|_v^{\mathfrak{M}} \to |B|_v^{\mathfrak{M}}$.[2]

DEFINITION 5. An arbitrary $L_{\to}$-formula A is a *tautologie* in $\mathfrak{M}$ iff $|A|_v^{\mathfrak{M}} \in D$ for every valuation v in $\mathfrak{M}$.

Let's formulate two versions of *modus ponens*:

(i) *stronger*: $\forall \mathfrak{M} \forall v[(|A|_v^{\mathfrak{M}} \in D \,\&\, |A \to B|_v^{\mathfrak{M}} \in D) \Rightarrow (|B|_v^{\mathfrak{M}} \in D)]$;

(ii) *weaker*: $\forall \mathfrak{M} [\forall v(|A|_v^{\mathfrak{M}} \in D) \,\&\, \forall v(|A \to B|_v^{\mathfrak{M}} \in D) \Rightarrow \forall v(|B|_v^{\mathfrak{M}} \in D)]$.

2.1 Modus ponens: two-valued and three-valued cases

Let $\mathfrak{M}_2$ be a two-valued logical matrix

$$\mathfrak{M}_2 =< \{1, 0\}, \to, \{1\} > .^3$$

[2]For the clarity we use the same symbols both for language functor (propositional connective) and corresponding matrix function.

[3]$\to$ is defined through the usual two-valued truth table.

It can be seen that in two-valued case *modus ponens* principle in form (i) and in form (ii) are equivalent.

But in case of three-valued logic we have another situation: (i) and (ii) are not equivalent, but (i) entails (ii), the reverse is not the case.

Thus consider for instance the three-valued logical matrix

$$\mathfrak{M}_3 =< \{1, 1/2, 0\}, \rightarrow, \{1, 1/2\} >,$$

where $\rightarrow$ is defined by the folowing truth-table:

$\rightarrow$	1	1/2	0
1	1	1	0
1/2	1	1	1/2
0	1	1	1

It can be easily seen that *modus ponens* in form (i) is not valid in the matrix $\mathfrak{M}_3$. That is, for some A and B there is valuation v such that $|A|_v = 1/2$, $|A \rightarrow B|_v = 1/2$ and $|B|_v = 0$.

Although *modus ponens* is not "normal" in the matrix $\mathfrak{M}_3$, it is a tautologousness-preserving rule of inference.

THEOREM 1. *Modus ponens in form (ii) is hold in the matrix $\mathfrak{M}_3$, that is*

$$\forall v(|A|_v \in \{1, 1/2\}) \,\&\, \forall v(|A \rightarrow B|_v \in \{1, 1/2\}) \Rightarrow \forall v(|B|_v \in \{1, 1/2\}).$$

PROOF.

1.	Let theorem does not hold	– assumption
2.	$\forall v(\lvert A\rvert_v \in \{1, 1/2\})$ and $\forall v(\lvert A \rightarrow B\rvert_v \in \{1, 1/2\})$ and $\exists v(\lvert B\rvert_v \notin \{1, 1/2\})$	– from 1
3.	$\forall v(\lvert A\rvert_v \in \{1, 1/2\})$	– from 2
4.	$\forall v(\lvert A \rightarrow B\rvert_v \in \{1, 1/2\})$	– from 2
5.	$\exists v(\lvert B\rvert_v \notin \{1, 1/2\})$	– from 2
6.	$\lvert B\rvert_{v'} = 0$	– from 5, elim. of quantifiers
7.	$\lvert A\rvert_{v'} \in \{1, 1/2\}$	– from 3, elim. of quantifier

Then we have two cases:

Case 1.

8.	$\lvert A\rvert_{v'} = 1$	– from 7

9. $|A|_{v'} \to |B|_{v'} = 0$ – from 8, 6, definition of $\to$
10. $|A \to B|_{v'} = 0$ – from 9, def. 4
11. $\exists v(|A \to B|_v \notin \{1, 1/2\})$ – from 10
12. Assumption (1) is incorrect – from 11, 4

Case 2.
13. $|A|_{v'} = 1/2$ – from 7
14. Let's define the following valuation v'' :

$$v''(p) = \begin{cases} v'(p), \text{if } v'(p) = \{1, 0\} \\ 1, \text{if } v'(p) = 1/2. \end{cases}$$

Then two lemmas on v' and v'' valuations take palace.

Lemma 1. $\forall A$: if $|A|_{v'} = 1/2$, then $|A|_{v''} = 1$.

Lemma 2. $\forall A$: if $|A|_{v'} = 0$, then $|A|_{v''} = 0$.

These short lemmas can be proved by induction on the structure of formula A.

Let's continue our proof of theorem:

15. $|A|_{v''} = 1$ – from 13 by lemma 1
16. $|B|_{v''} = 0$ – from 6 by lemma 2
17. $|A|_{v''} \to |B|_{v''} = 0$ – from 15, 16, definition of $\to$
18. $|A \to B|_{v''} = 0$ – from 17, definition 4
19. $\exists v(|A \to B|_v \notin \{1, 1/2\})$ – from 18
20. Assumption (1) is incorrect – from 19, 4

Thus the theorem is proved. □

So we have considered two nonequivalent versions of *modus ponens* in terms of logical matrix $\mathfrak{M}_3$.

3 Natural implication: extended class

Let's recall the definition of *natural* implication, which was introduced in [7]:

Definition 6. Implication is called *natural* if it satisfies the following criteria:

(1) **C**-extending, i.e. restrictions to the subset $\{0, 1\}$ of V_3 coincide with the classical implication.

(2) If $p \to q \in D$ and $p \in D$, then $q \in D$, i.e. matrices for implication need to be normal in the sense of Łukasiewicz-Tarski (they verify the modus ponens) [2, p. 134].

(3) Let $p \leq q$, then $p \to q \in D$.

(4) $p \to q \in V_3$, in other cases.

Under the definition 6 there are 6 implications when $D = \{1\}$ and 24 implications when $D = \{1, 1/2\}$.

So, according to previously reviewed two versions of *modus ponens* principle we can transform our definition of *natural* implication and weaken the condition of normality, i.e. condition (2) can be replaced by *modus ponens* principle in form (ii).

As the result, we obtain a new update definition of *natural* implication and respectively extended class of implications, which satisfy that transformed criteria.

Thus, there are 8 implications when $D = \{1\}$:

$\to$	1	1/2	0
1	1	a	0
1/2	1	1	b
0	1	1	1

$\to$	1	1/2	0
1	1	1	0
1/2	1	1	c
0	1	1	1

where $a \in \{0, 1/2\}$, $b \in \{0, 1/2, 1\}$ and $c \in \{1, 0\}$.

When $D = \{1, 1/2\}$ the class of natural implications is expanded significantly. In this case there are 72 implications:

$\to$	1	1/2	0
1	1	b	0
1/2	a	a	b
0	1	a	1

where $a \in \{1, 1/2\}$ and $b \in \{0, 1/2, 1\}$.

So, the extended class of natural implications consists of 80 implications, which defined by 72 truth tables mentioned above. (Note that 8 truth tables with $D = \{1\}$ are included in the list of truth tables with $D = \{1, 1/2\}$.)

The fact that *modus ponens* principle in form (ii) holds in the matrices with the *natural* implications, defined by the foregoing truth tabeles, can be proved the same way (with some modifications) as in section 2.1.[4]

[4]The proof is required for the implications, for which *modus ponens* in form (i) is not hold.

So, as the result of the trasformation of the definition of natural implication in terms of tautologousness-preserving property of *modus ponens*, we have 44 new truth tables for natural implications, which are not appeared while using the unchanged definition.

More examples of three-valued implication, which satisfy only the weak formulation of *modus ponens* can be found in papers [4, 3]. In both cases the implications verify all tautologies of classical propositional logic. The complete list of truth tables for such implications can be found in [1, p. 64, 80–81]. Our extended class of natural implications is different in that not all implications verify all tautologies of classical propositional logic.

It is of interest to continiue our investigations of functional properties of three-valued logics (q.v. [7]) and consider the extensions of regular Kleene's logics by these new implications. It is expected that in our classification of three-valued logics will appear some new basic systems.[5]

References

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[5] In [7], on examination of the implicative extensions of weak Kleene's logic we received 7 *basic* logics: Łukasiewicz's logic $\mathbf{Ł_3}$, paraconsistent logic **PCont**, three-valued Bochvar's logic $\mathbf{B_3}$, logic $\mathbf{Z}$, $\mathbf{T^3}$, $\mathbf{T^2}$ and $\mathbf{T^1}$, which form a lattice w.r.t. relation of functional inclusion one logic to another.